**The Metric**

Spacetime and Geometry : An Introduction to General Relativity – by Sean M Carroll

By George Keeling, created 3 May 2019, heavily updated July 2020

I started to write this in May 2019 when I read section 2.5 and it has remained a work in progress until now. I have quite a collection of metrics: 3D Euclidean, Plane polar, Spherical polar, Surface of sphere (S2), Minkowski, Spherical Minkowski, Schwarzschild, Eddington-Finkelstein, Kruskal. I will add more as I find them.

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We start with a bit of general information about a metric, then list each of the metrics, then have a few examples on spherical metrics and proper time.

At the beginning of section 2.5 Carroll writes that the metric is denoted by is a symmetric tensor and we can usually define its inverse by

|  |  |
| --- | --- |
|  | (1) |

Carroll uses the symbol g but I don't know how to get that in Microsoft equations so I always use .

He then lists various facts about the "glory" of the metric which are worth repeating. They are

1. The metric supplies a notion of "past" and "future".
2. The metric allows the computation of path length and proper time.
3. The metric determines the "shortest distance" between two points, and therefore the motion of test particles.
4. The metric replaces the Newtonian gravitational field .
5. The metric provides a notion of locally inertial frames and therefore a sense of "no rotation".
6. The metric determines causality, by defining the speed of light faster than which no signal can travel.
7. The metric replaces the traditional Euclidean three-dimensional dot product of Newtonian mechanics.
8. The (inverse) metric lowers (raises) indices:

I added the last one. It is related to the second last.

## Representation of the metric

Using spherical coordinates as an example, the metric is usually represented as an equation like

|  |  |
| --- | --- |
|  | (2) |

where is the invariant line element (number 2 on the glorious list) or sometimes as a matrix like

|  |  |
| --- | --- |
|  | (3) |

Why are they equivalent? The answer is very simple. (2) can be written

|  |  |
| --- | --- |
|  | (4) |

so the components of given at (3) are the coefficients in the equation given at (2).

I try to refer to an equation like (2) as the 'metric equation'. Most people just call it the 'metric'. The form as in (2) has two great advantages. It is compact and it immediately tells you what all the coordinates are.

Then we come to the vexed question of why we are using and not . The latter is frequently used (and much easier to write especially in latex) but is not strictly correct. is a basis dual vector (or basis one form) and is some other sort of number. In particular when , but . This can affect calculations on the metric equation and for example we will meet the Eddington-Finkelstein metric whose equation is strictly

|  |  |
| --- | --- |
|  | (5) |

It could be sloppily written

|  |  |
| --- | --- |
|  | (6) |

Since the metric is always symmetric in practice it doe not make much difference.

Also we frequently suddenly change from using the strict to the easy . There is an example later when we look at proper time.

February 2022 are learn more about tensors and in fact are rank (0,2) basis tensors. See [Blennow basis basics.docx](https://docs.google.com/document/d/18rQwr292hh55DwXNfeqBzqUlhh1jXItg?rtpof=true).

## Metric or coordinates?

We often change coordinates to map the same space or spacetime. The most familiar example is 2D Euclidean and polar coordinates. We then get two metric equations

|  |  |
| --- | --- |
|  | (7) |

I would talk about the first being the Euclidean metric and the second the polar metric. I think that is strictly incorrect. They are the same metric in different coordinates, sort of.

# Two and three dimensional geometry metrics

These are called Euclidean or Riemannian metrics.

## 3D Euclidean

|  |  |
| --- | --- |
|  | (8) |
|  | (9) |

This is the one that we are all used to. It is so simple that it is difficult to understand what metrics are all about.

## Plane polar coordinates

Coordinates radial, angle to axis

|  |  |
| --- | --- |
|  | (10) |

We often index tensor components by the coordinate. So in this case . because the metric is always symmetric.

## Spherical polar

Coordinates radial, polar, azimuthal (= longitude)

|  |  |
| --- | --- |
|  | (11) |

## 2-sphere: Surface of sphere (S2)

Coordinates polar, azimuthal

|  |  |
| --- | --- |
|  | (12) |

This metric is often written

|  |  |
| --- | --- |
|  | (13) |

So is the angle subtended between two points.

## Ellipsoid, Elliptic paraboloid, Hyperbolic paraboloid

In [Commentary 8 Curvatures 2D.pdf](https://drive.google.com/open?id=1r_vRryZ9BM_l26-qpqFh2Rr-NniFYLVM) and [Commentary 8 Curvatures 2D calculations.pdf](https://drive.google.com/open?id=1qAmOlFdRNPM49-71ex8rlUCpWxr5OnQf) I investigate more 2D surfaces and have metrics, Riemann tensors and curvatures for the following, being constants,

Ellipsoid coordinates

|  |  |
| --- | --- |
|  | (14) |

Elliptic paraboloid coordinates

|  |  |
| --- | --- |
|  | (15) |

Hyperbolic paraboloid coordinates

|  |  |
| --- | --- |
|  | (16) |

## General 2D case

with coordinates . Obviously there is no particular metric so we just say it is

|  |  |
| --- | --- |
|  | (17) |

The Riemann tensor has only one independent component. It is convenient to use for that. In all up to eight of the possible sixteen are non zero. The Ricci tensor is

|  |  |
| --- | --- |
|  | (18) |

or

|  |  |
| --- | --- |
|  | (19) |

and the scalar curvature is

|  |  |
| --- | --- |
|  | (20) |

## Poincare half plane

The metric of the Poincare half plane is

|  |  |
| --- | --- |
|  | (21) |

Carroll gives it at his equation 3.192. It is interesting because it has constant negative curvature in contrast to S² which has constant positive curvature. They are both maximally symmetric. Carroll says that it is a hyperboloid H² and cannot be isometrically embedded in **R**³. Wikipedia gives another definition of a hyperboloid.

See [Commentary 8 Curvatures 2D More.docx](https://drive.google.com/open?id=1GqHUfpIPA6kAECISjjPe3J2QhyUJZrWh).

The meaning of isometric embedding becomes clearer in the discussion on tori in [Physics Forums](https://www.physicsforums.com/threads/is-a-three-torus-maximally-symmetric-does-curvature-determine-volume.996651/post-6424208).

## Two torus T²

There are two kinds of tori flat and curved. They are both generated by two circles but to find the former you need to start in four dimensions. A metric for a curved torus is

|  |  |
| --- | --- |
|  | (22a) |

and for a flat torus

|  |  |
| --- | --- |
|  | (23a) |

In both cases are the radii of the generating circles. See [Commentary 8.2 Tori.pdf](https://drive.google.com/open?id=1Jq50s3ojRV7CBjqyb2QzP87JYLv4tJIG). The flat torus is notable because it is flat but has a finite area.

## 3-sphere

Coordinates , metric

|  |  |
| --- | --- |
|  | (22) |

See <https://en.wikipedia.org/wiki/3-sphere> and [Ex 3.08 Vital statistics of a 3-sphere.pdf](https://drive.google.com/open?id=1wYyRaiV67uIT0I-sPxJdSAJL8pBskiYr) for Riemann tensor and more.

# Relativistic metrics

These are called Lorentzian or pseudo-Riemannian metrics.

In special and general relativity we add a zeroth time dimension / coordinate (or ) but it is mighty peculiar because the (or ) component of the metric is negative and the is multiplied by the speed of light which leads to all the amazing results. Also, because the speed of light is so huge, it took a long time (from Galileo to Einstein) for anybody to notice this.

So we write the metric as

|  |  |
| --- | --- |
|  | (23) |

This is called a signature and the speed of light is conveniently set to 1. is the line element and its negative is proper time. The proper time is the time measured on a clock that is stationary with respect to an observer. (Since practically everything is nearly stationary compared to the speed of light, and are indistinguishable in everyday life.)

## Minkowski

Coordinates

|  |  |  |
| --- | --- | --- |
|  |  | (24) |

## Spherical Minkowski

Coordinates as in spherical

|  |  |
| --- | --- |
|  | (25) |

We could also write

|  |  |
| --- | --- |
|  | (26) |

## Schwarzschild metric (spherical polar)

See Carroll's 5.1. Coordinates are time, radial, polar, azimuthal

|  |  |
| --- | --- |
|  | (27) |

We often replace (twice Newton's gravitational constant times the central mass) by the Schwarzschild radius which is the radius of event horizon and the last part by . As matrices we have

|  |  |  |
| --- | --- | --- |
|  |  | (28) |
|  |  | (29) |

The function often has amazing self-cancelling properties under operations such as differentiation.

### Null, timelike and spacelike coordinates

This is the first time we meet a metric where some of the diagonal components vanish. In this case the component vanishes on the event horizon at . The component is worse, it goes to infinity. Inside the event horizon is positive and is negative. A coordinate that has negative diagonal metric component is called timelike, zero null and positive spacelike. So inside is timelike and decreasing is like time ticking forward which is why, once inside the event horizon, the singularity at is the inevitable destination. (I'm not sure why decreasing rather than increasing is getting 'older').

Carroll says in section 2.5 on the metric on p 73 "if the metric is continuous and nondegenerate, its signature will be the same at every point- We will always deal with continuous, nondegenerate metrics." I'm not how that works with the Schwarzschild metric.

The answer to this came in a discussion on basis vectors on Physics forums. The coordinates inside the event horizon are not the same as those outside. The two regions are separated by the event horizon. They are not connected! The direction of time inside the event horizon is arbitrary. If getting older implies decreasing then, you are in the black hole, if it implies increasing then you are in the white hole. So we really should have three sets of coordinates: , and for normal space, in the black hole and in the white hole. One max need a fourth for region of a Kruskal diagram.

On that thread there was also a great discussion null, timelike and spacelike coordinates and how to characterise them. I discuss it all in 'Commentary 2.3 Coordinates and basis vectors'.

## Eddington-Finkelstein metric

The Eddington-Finkelstein metric is for the same spacetime as Schwarzschild but with coordinates

|  |  |
| --- | --- |
|  | (30) |

where

|  |  |
| --- | --- |
|  | (31) |

if we also add

|  |  |
| --- | --- |
|  | (32) |

we can get a metric equation with coordinates

|  |  |
| --- | --- |
|  | (33) |

with implicitly defined by

|  |  |
| --- | --- |
|  | (34) |

It looks like is null in (30), is timelike, null, then spacelike as we move towards .

## Kruskal predecessor

On the way to the Kruskal metric we get coordinates with metric equation

|  |  |
| --- | --- |
|  | (35) |

where is implicitly defined in terms of as

|  |  |
| --- | --- |
|  | (36) |

In terms of the coordinates at (31),(32) we also have

|  |  |
| --- | --- |
|  | (37) |

I haven't seen this formulation with the sign inside the event horizon elsewhere but Carrol implicitly refers to it. I proved it in 'Commentary 5.7 Kruskal coordinates'.

In this coordinate system and are both always null.

## Kruskal

Kruskal coordinates are where

|  |  |
| --- | --- |
|  | (38) |
|  | (39) |

and they give a metric equation

|  |  |
| --- | --- |
|  | (40) |

with implicitly defined from

|  |  |
| --- | --- |
|  | (41) |

It is useful to have eliminated from the solution because it is greater than infinity inside the event horizon!

These coordinates have great advantages because they cover the whole of spacetime very regularly. is timelike everywhere and is spacelike everywhere and there are other similarities with flat Minkowski spacetime.

## de Sitter x 4

de Sitter metrics are maximally symmetric and therefore have constant curvature. The first de Sitter metric has positive curvature:

|  |  |
| --- | --- |
|  | (42) |

de Sitter conformal

|  |  |
| --- | --- |
|  | (43) |

Anti de Sitter spacetime has negative curvature and its metric is

|  |  |
| --- | --- |
|  | (44) |

Anti de Sitter conformal

|  |  |
| --- | --- |
|  | (45) |

Carroll describes these in section 8.1 and I investigated at

[Commentary 8.1 Maximally symmetric universes.pdf](https://drive.google.com/open?id=1p5-qxrQ6FsuOJ3APTuflONutKefzwP-6) where among other things I calculate the Anti de Sitter conformal Riemann, Ricci tensors and .

## Robertson-Walker (FLRW)

Other names Friedmann, Lemaitre. Carroll's equations 8.38, 8.43

|  |  |
| --- | --- |
|  | (45a) |

The second version is Carroll's preferred form - 'flouting' conventional wisdom.

## Many, many more

Catalogue of space times from Minkowski to Sultana-Dyer and more by Thomas Müller and Frank Grave at <https://arxiv.org/pdf/0904.4184v3.pdf>

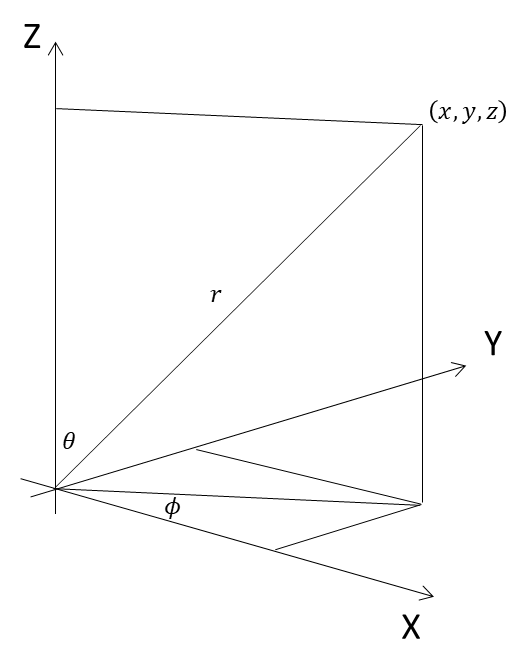
Also on my drive: [Catalogue\_of\_Spacetimes.pdf](https://drive.google.com/open?id=1qjjN011zV0PPO1U-HQGzo1B2tButPYZQ)

# Three examples

## Metric in spherical coordinates

The coordinates radial, polar, azimuthal (= longitude) as shown below. When I started on Carroll's book I had the coordinates the other way round. Unfortunately there are many conventions for these coordinates. I now stick to the one used here, which is the 'physicist convention' and ISO 31-11.

We'll calculate the metric again in spherical coordinates using the simple minded way (as Carroll calls it at the end of appendix A):



The mapping from polar to cartesian coordinates is

|  |  |
| --- | --- |
|  | (46) |
|  | (47) |
|  | (48) |

The cartesian metric is

|  |  |
| --- | --- |
|  | (49) |

We turn (46) into infinitesimals and it becomes

|  |  |
| --- | --- |
|  | (50) |

The way I justify this is by imagining differentiating (46) with respect to another variable say and we would get

|  |  |
| --- | --- |
|  | (51) |
|  | (52) |

Now do the easy derivatives then multiply the whole equation by and we have (50).

Similarly we get

|  |  |
| --- | --- |
|  | (53) |
|  | (54) |

This might be called 'implicit differentiation'. It's really just exercising the Leibniz and chain rules. Also remember that that we should have been writing etc really. So in this case we will not assume that etc.

Now stick expressions for into (49) and we get something that gets worse before it gets better:

|  |  |  |
| --- | --- | --- |
|  |  | (55) |
|  |  |  |
|  |  |  |
|  |  | (56) |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  | (57) |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  |  |
|  |  | (58) |

In (56) we multiplied out the squares. In (57) we grouped all the terms by . For (58) we factorised and repeatedly used . We were always careful not to abuse the order of . And we have found the correct answer.

The tensor transformation law (or the pullback matrix which Carroll discusses in Appendix A) gives us

|  |  |
| --- | --- |
|  | (59) |

In this case the primed system is the spherical coordinate system and the unprimed flat Euclidean and the identity, which considerably simplifies the summation in (59) .

from (46)-(48)

|  |  |
| --- | --- |
|  | (60) |
|  | (61) |
|  | (62) |

Using those in (59) we get

|  |  |  |
| --- | --- | --- |
|  |  | (63) |
|  |  | (64) |
|  |  | (65) |
|  |  | (66) |
|  |  | (67) |
|  |  | (68) |
|  |  | (69) |
|  |  | (70) |
|  |  | (71) |

We have used the symmetry of the metric, In matrix form that is

|  |  |
| --- | --- |
|  | (72) |

That was a good deal easier than the 'simple minded way' and gives the same answer.

## Calculate S2

As an afterthought here we can write down the metric for the surface of a sphere of unit radius. Setting and then removing that coordinate, (2) and (3) become

|  |  |
| --- | --- |
|  | (73) |
|  | (74) |

with .

## Four velocities

The general expression for the relativistic metric

|  |  |
| --- | --- |
|  | (75) |

gives us an interesting fact about 4-velocities which are defined as

|  |  |
| --- | --- |
|  | (76) |

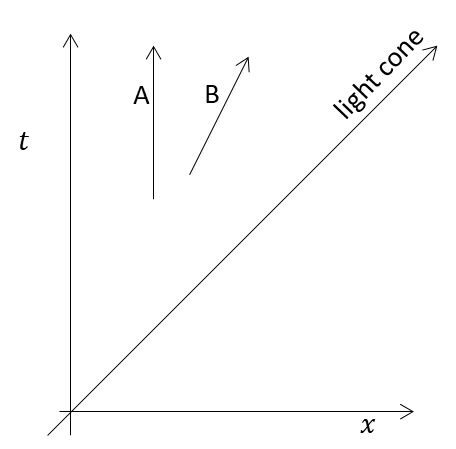
The norm of the four velocity is

|  |  |
| --- | --- |
|  | (77) |

We get the by simply dividing (75) by and italicising 's.

A long long time ago I got confused by four velocities. Here's what I did which may be worth a look.

Let's consider two particles A and B in flat space time on the diagram below. A is stationary, B is moving at half the speed of light (equation of path .



For A we might say that

|  |  |
| --- | --- |
|  | (78) |
|  | (79) |

(77) becomes

|  |  |
| --- | --- |
|  | (80) |

Correct!

For B we might say

|  |  |
| --- | --- |
|  | (81) |
|  | (82) |

(77) becomes

|  |  |
| --- | --- |
|  | (83) |

Oops!

That's because we have parameterised the path wrongly ( looks very odd) and we should not assume that the parameter is proper time. (81) and (82) should be

|  |  |
| --- | --- |
|  | (84) |
|  | (85) |

(83) becomes

|  |  |
| --- | --- |
|  | (86) |

but we have the metric (23)

|  |  |
| --- | --- |
|  | (87) |

therefore

|  |  |
| --- | --- |
|  | (88) |

using (84)

|  |  |
| --- | --- |
|  | (89) |

which perfectly cancels the 3 in (86).

Integrating (89) we get

|  |  |
| --- | --- |
|  | (90) |

So with suitable initial conditions setting (84) should have been

|  |  |
| --- | --- |
|  | (91) |

The parameterisation of A was 'lucky'. If we had said at (78) we would have got the wrong answer. We know that for a particle at rest which was why we were lucky.

This just shows that you can't use proper time carelessly as a parameter in equations.

# References

Blog: <https://www.general-relativity.net/>

Docx file: [Commentary 2.5 The Metric.docx](https://drive.google.com/open?id=1N61ui8BuC4CGRwWNK0UCi8hldhFjfqQX)

Pptx file: [Commentary 2.5 The Metric.pptx](https://drive.google.com/open?id=19H0-Mdpch9iwW9k_Q1wEUYB33RVRNBMN)

The fab four

[Commentary Important Equations.pdf](https://drive.google.com/open?id=1GWyXlMSyBsBRv2mVns3HQsGZ0-XWDidt)

[Commentary 2.5 The Metric.pdf](https://drive.google.com/open?id=1GMA1VfrvDBogsYEutMCxZTgK6h1chWjz)

[Commentary Tensor Tricks.pdf](https://drive.google.com/open?id=1Zy60wMNwbFowwSF31X6mDnupEOWtC1kI)

[Commentary Constants and conversion factors.pdf](https://drive.google.com/open?id=1AQJtesVCZMp8C2nRhhABUWE7SGrYmSki)

Others

[Commentary 2.3 Coordinates and basis vectors.pdf](https://drive.google.com/open?id=1qYpubQmANO0Np_ETMf0U7F837LKDv__f)

[Ex 3.06 Metric outside Earth.pdf](https://drive.google.com/open?id=1aumAWmQdB7t0n0LV1iLdcffOKLVouFjw)

[Commentary 3 Near earth metric and Christoffel symbols.docx](https://drive.google.com/open?id=1yfnamxaqTunz1vyOF2VDe9H-qXRgt-Cq)

[Commentary 5.7 Kruskal coordinates.pdf](https://drive.google.com/open?id=1IxnQrwK9K8bCYWkpKzQYX9dgBOQ3HiNJ)

[Commentary 8 Curvatures 2D.pdf](https://drive.google.com/open?id=1r_vRryZ9BM_l26-qpqFh2Rr-NniFYLVM)

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<https://en.wikipedia.org/wiki/Metric_tensor>

Spherical coordinates conventions

<https://mathworld.wolfram.com/SphericalCoordinates.html>

ISO 31-11 <https://en.wikipedia.org/wiki/ISO_31-11>

Catalogue of space times <https://arxiv.org/pdf/0904.4184v3.pdf>

Also on my drive: [Catalogue\_of\_Spacetimes.pdf](https://drive.google.com/open?id=1qjjN011zV0PPO1U-HQGzo1B2tButPYZQ)